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ON THE GENERALIZED OVERRELAXATION
METHOD FOR OPERATOR EQUATIONS

by

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June 14, 1962

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ERRATA

NYO-9192

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Please note the following changes:

Page 3 - Section 3 should read ....The Main Theorem

Page 4 - The third line in the second paragraph should read
.....(go-method)...

Page 5 - The twelfth line should read.....(go-method)....

Page 18 - The first line of the third paragraph should read
.....(go-method)....

Page 6 - The first line should read ....Lemma 1......

Page 9 - The third line should read.....Lemma 1......

Page 6 - The lower case w in equation no. 9 should
read ω

Page 8 - The lower case w's on lines 8, 14 and 18
should read ω

Page 10- The lower case w on line 9 should read ω

Page 13- The lower case w's on lines 6 and 12 should read ω

Page 12- The equation on line 12 should read

$$\|(T-\lambda)u\| \geq \Theta\|u\|$$

ABSTRACT

In this article we derive the necessary and sufficient
conditions for the spectrum of the generalized overrelaxation
operator

$$T(\omega) = (D+\omega S)^{-1} \left\{ (\omega-1)D + \omega Q \right\}$$

to lie in the interior of the unit circle centered at the origin.

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ON THE GENERALIZED OVERRELAXATION METHOD
FOR OPERATOR EQUATIONS*

W. V. Petryshyn†

1. Introduction.

In [1] Householder, using Weissinger's identity, obtained the necessary and sufficient conditions for the convergence of the Seidel method for the solution of finite matrix equations of the form $(D-S-S^*-F)u = f$. The same conditions were also obtained by Krein and Prozorovskaya [2] for an analog of the Seidel method for the operator equations of the form $(D+S+S^*)u = f$ in a Hilbert space.

The purpose of this article is to extend the result of the above authors to the generalized overrelaxation iterative method (goi-method) for the solution of a wider class of operator equations in a Hilbert space investigated by the author [3]. It includes as special cases the corresponding results given in [1,2,3].

2. The Identities.

Let H be a real or complex Hilbert space and A a linear bounded operator in H of the form

(1) $$A = D + S + Q.$$

Let S^* be the adjoint of S and Ω a set of real positive numbers $\omega > 0$ such that Ω and the operators $D, S,$ and Q

have the proper y that there exists some linear bounded operator K such that

(a) $(Du, Kv) = (Ku, Dv)$ for all u and v in H, i.e. D is K-symmetric

(b) $G = G(\omega) = \frac{2-\omega}{\omega} D + S^* - Q$ is K-symmetric and K-positive definite (K-p.d.), i.e., there exists a real number $\beta = \beta(\omega) > 0$ such that for all $u \in H$ and $\omega \in \Omega$

(2)
$$(Gu, Ku) \geq \beta \|u\|^2 .$$

(c) $(D+\omega S)$ has a bounded inverse defined on all of H.

In seeking an approximate solution u_n of the equation

(3)
$$A\ u = f\ ,\qquad\qquad f \in H\ ,$$

we shall apply the goi-method with a constant relaxation factor ω in Ω defined by the scheme

(4)
$$(D+\omega S)u_n = -\left\{(\omega-1)D + \omega Q\right\} u_{n-1} + \omega\ f\ ,$$

or equivalently by

(5)
$$u_n = -\ T(\omega)u_{n-1} + g\ ,$$

where u_0 is an arbitrary initial approximation, $g = \omega(D+\omega S)^{-1}f$, and

(6)
$$T(\omega) = (D+\omega S)^{-1}\left\{(\omega-1)D + \omega Q\right\}\quad .$$

Lemma. Let D, S, Q, and Ω satisfy the conditions (a), (b) and (c) in which K is symmetric, positive definite, and commutes with S. Then for every u and v in H the following identities are valid

$$(7) \quad (Ku, Av) = \sum_{i=0}^{n-1} (KT^i[I+T]u, GT^i[I+T]v) + R_n\{u, v\}$$

$$(8) \quad |([D+\omega S]u, Ku)|^2 - |([(\omega-1)D+\omega Q]u, Ku)|^2 = \omega^2(Gu, Ku)(Au, Ku),$$

where

$$(9) \quad R_n\{u, v\} = \omega^{-1}(T^n u, K[D+\omega S]T^n(I+T)v), \quad n = 1, 2, 3, \ldots .$$

Proof. Before we proceed with the proof of (7) and (8) which requires some manipulation let us first observe that since K is symmetric the condition (a) implies that

$$(10) \qquad\qquad D*K = K D$$

while this together with (b) and the commutativity of K with S implies that

$$(11) \qquad\qquad KS - Q*K = KS* - KQ .$$

Furthermore, (10) and (11) together imply that A is K-symmetric. On the other hand if conditions (a) and (b) are satisfied and A is K-symmetric, then K commutes with $S+S*$. In fact (a) and (b) imply that $SK + KQ = KS* + Q*K$ while (a) and K-symmetry of A imply that $KS + KQ = S*K + Q*K$. Together they show that $(S*+S)K = K(S*+S)$, i.e., K commutes with $S+S*$.

We shall now prove the identity (7) by using the mathematical

induction according to which we must first verify its validity
for $n = 1$. If u and v are arbitrary elements in H, then
by (6)

(12) $[(\omega-1)D + \omega Q]u = (D+\omega S)\ Tu$

or

(12') $([(\omega-1)D+\omega Q]u, Kv) = ([D+\omega S]Tu, Kv)$

whence, using (10), we obtain

$$(u, [(\omega-1)KD+\omega Q^*K]v) = (Tu, [KD+\omega S^*K]v).$$

Adding and subtracting $(Tu, K[(\omega-1)D+\omega Q]v)$ on the right and
using the commutativity of K with S we get

$$(u, [(\omega-1)KD+\omega Q^*K]v) = (Tu, \omega KGv) + (Tu, K[(\omega-1)D+\omega Q]v)$$

or, after adding $(u, \omega KGv)$ to both sides and cancelling,

$$(u, [KD+\omega Q^*K+\omega K(S^*-Q)]v) = ([I+T]u, \omega KGv) + (Tu, K[(\omega-1)D+\omega Q]v).$$

In view of (11) and (12), the last identity can be written as

(13) $(Ku, [D+\omega S]v) = ([I+T]u, \omega KGv) + (Tu, K[D+\omega S]Tv).$

Since, by (12), $\omega Au = D+\omega S + (\omega-1)D + \omega Q = (D+\omega S)u + (D+\omega S)Tu$
we obtain from it and (13)

$$(Ku, \omega Av) = (Ku, (D+\omega S)v) + (Ku, (D+\omega S)Tv) =$$
$$= ((I+T)u, \omega KGv) + (Tu, K(D+\omega S)Tv) +$$
$$+ ((I+T)u, \omega KGTv) + (Tu, K(D+\omega S)T\ Tv)$$
$$= \omega((I+T)u, KG(I+T)v) + (Tu, K(D+\omega S)\ T(1+T)v)$$

On dividing by ω we get (7) for $n = 1$.

Let us assume now that (7) is valid for $n = k$, i.e.,

$$(7') \quad (Ku, Av) = \sum_{i=0}^{k-1} (KT^i(I+T)u, GT^i(I+T)v) + R_k\{u, v\},$$

and prove its validity also for $n = k+1$. This, however, follows from $(7')$, (13), and (9) for if in (13) we replace u by $T^n u$ and v by $T^n(I+T)v$, then from (9) we get

$$R_k\{u, v\} = \omega^{-1}((I+T)T^n u, \omega KGT^n(I+T)v)$$
$$+ w^{-1}(T^{n+1}u, K(D+wS)T^{n+1}(I+T)v)$$

$$= ((I+T)T^n u, KGT^n(I+T)v) + R_{k+1}\{u, v\}.$$

Combining this with $(7')$ we get the validity of (7) for $n = k+1$ and thus prove the first identity.

To prove the second identity (8) note that for every $u \in H$

$$\omega^2(Gu, Ku)(Au, Ku) = \omega(Gu, Ku)((D+\omega S)u, Ku) + \omega(Gu, Ku)(((\omega-1)D+\omega Q)u, Ku).$$

Adding $([D+wS^*-\omega G]u, Ku) \cdot ((D+wS)u, Ku)$ and subtracting its equivalent $([(\omega-1)D+\omega Q]u, Ku) \cdot ((D+wS)u, Ku)$ on the right we get

$$\omega^2(Gu, Ku)(Au, Ku) = ((D+\omega S^*)u, Ku)\ ((D+\omega S)u, Ku)$$
$$- ([(\omega-1)D+\omega Q]u, Ku)\ ([D+\omega S-\omega G]u, Ku)$$
$$= |((D+wS)u, Ku)|^2 - ([(\omega-1)D+\omega Q]u, Ku)([(\omega-1)D+\omega S+\omega(Q-S^*)u, Ku)$$

for, as is easily seen, $\overline{((D+\omega S)u, Ku)} = ((D+\omega S^*)u, Ku)$.

Since K is symmetric, the relation (11) implies that the last identity can be written in the form

$$\omega^2(Gu, Ku)(Au, Ku) = |((D+\omega S)u, Ku)|^2$$
$$- ([(\omega-1)D+\omega Q]u, Ku)([(\omega-1)D+\omega K^{-1}Q^*K]u, Ku).$$

Since $\overline{([(\omega-1)D+\omega Q]u,Ku)} = (Ku,[(\omega-1)D+\omega Q]u) = ([(\omega-1)KD+\omega Q^*K]u,u)$
$= ([(\omega-1D+\omega K^{-1}Q^*K]u,Ku)$, the last relation is exactly the identity
(8). This completes the proof of the Lemma.

Remark. If we choose $K = I$, $Q = S^*$, and $\omega = 1$, then the
identity (7) reduces to the identity used in [2]. If in this
case we, in addition, assume that **A** is symmetric and positive
definite, then from the identity (8) valid for all u in H we
obtain the inequality used in [2].

   In Lemma 2 below we will establish the relationship between
the identity (8) and the important equality (21) in [3] and in
this case also the relationship between (7) and (8).

Lemma 2. If in addition $T(\omega)$ satisfies the condition of Theorem 1
in [3], i.e., $\sigma(T)$ contains only eigenvalues μ of finite
multiplicity with zero as its sole limit point, then the identity
(7) reduces to the equality
(7')
$$(Ku,Au) = |1+\mu|^2(Ku,Gu) \sum_{i=0}^{n-1} |\mu|^{2i} + \omega^{-1} |\mu|^{2n}(1+\overline{\mu})(Ku,[D+\omega S]u)$$

and (8) to the equality (21) in [3] which for **A** of the form (1)
can be written as

(8') $(Gu,Ku) = \dfrac{1-|\mu|^2}{|1+\mu|^2} (Au,Ku)$,

where μ is an arbitrary eigenvalue of T and $u \neq 0$ an
eigenvector corresponding to μ. Furthermore, if **A** is also
K-p.d. then for $n \rightarrow \infty$ the equality (7') is identical with
(8').

Proof. The proof of (7') follows directly from (7) with v = u
and the observation that the equality $Tu = \mu u$ implies that
$T^i(I+T)u = \mu^i(1+\mu)u$ for i = 0,1,2,... , and
$(KT^i(I+T)u, GT^i(I+T)u) = |1+\mu|^2 |\mu|^{2i}(Ku, Gu)$.

   To prove (8') note that, in view of (6) and (12), $Tu = \mu u$
imples that

$$[(\omega-1)D+\omega Q]u = (D+\omega S)Tu = \mu(D+\omega S)u \quad .$$

Substituting this into the identity (8) we get

$$\omega^2(Gu,Ku) (Au,Ku) = [1-|\mu|^2] |((D+wS)u,Ku)|^2 .$$

Noting that $T = (D+\omega S)^{-1}\{(\omega-1)D+\omega Q\} = \omega(D+\omega S)^{-1}A - I$
and hence that -1 is not an eigenvalue of T we obtain
the equality $(D+\omega S)u = \frac{\omega}{1+\mu} Au$ which is valid for all eigen-
vectors u and corresponding eigenvalues μ of T. Now
substituting $\frac{\omega}{1+\mu} Au$ for $(D+\omega S)u$ in the right side of the
last equality and cancelling we obtain the desired equality (8').

   The validity of the last assertion in Lemma 2 follows from
(8') and (7') for since G and **A** are positive definite the
relation (8') implies that $|\mu| < 1$ and therefore passing to
the limit in (7') as $n \longrightarrow \infty$ we obtain

$$(Ku,Au) = |1+\mu|^2 (Ku,Gu) \sum_{i=0}^{\infty} |\mu|^{2i} = \frac{|1+\mu|^2}{1-|\mu|^2} (Ku,Gu)$$

which is the equality (8').

3.   The Main Theorem. Let us note that from (5) we find by
induction that

$$(14) \qquad u_n = \sum_{i=0}^{n-1} T^i g + T^n u_0$$

from which we see that the sequence of approximations u_n converges if the series $\sum_{i=0}^{\infty} T^i$ converges. The latter converges if the spectrum $\sigma(T)$ of T lies in the interior of the unit circle. We shall now prove the main

Theorem. If D, S, Q, and Ω satisfy the conditions (a), (b), and (c), and K is symmetric, positive definite, and commutes with S, then the necessary and sufficient condition that the spectrum $\sigma(T)$ lie in the interior of the unit circle is that the operator A be K-p.d.

Proof. The proof of the theorem is essentially based on the identities proved above.

Necessity. To prove it note that if $\sigma(T)$ lies in the interior of the unit circle, then it is not hard to see that $|R_n\{u,v\}| \to 0$ as $n \to \infty$ and the identity (7) implies that

$$(15) \qquad (Ku, Av) = \sum_{i=0}^{\infty} (K\, T^i(I+T)u,\; GT^i(I+T)v)$$

for all u and v in H. Thus, if in (15) we put $u = v$,

then, in view of condition (b), (Au,Ku) is a convergent series of positive numbers. In particular, we have

$$(Ku,Au) \geq (K(1+T)u, \; G(1+T)u) \; .$$

Since $\lambda = -1$ does not belong to $\sigma(T)$, the last inequality shows that A is K-p.d. and thus proves the necessity.

Sufficiency. To prove the sufficiency we must show that the assumption that A is K-p.d., i.e., there exists an $\alpha > 0$ such that for all u in H

(16)
$$(Au,Ku) \geq \alpha \|u\|^2 \; ,$$

implies that $(T-\lambda)$ is a continuously invertible operator for all $|\lambda| \geq 1$, i.e., $R(T-\lambda) = H$ and there exists an $\Theta > 0$ such that $|(T-\lambda)u| \geq \Theta|u|$ for all $u \, \varepsilon \, H$.

If $(T-\lambda)$ were not continuously invertible there would exist a sequence $\{u_n\}$, $\|u_n\| = 1$, such that $v_n = (T-\lambda)u_n \to 0$ as $n \to \infty$. Applying to v_n the operator $(D+\omega S)$ we get $(D+\omega S)v_n \to 0$, i.e.,

(17)
$$\{(\omega-1)D + \omega Q\} u_n - \lambda(D+\omega S)u_n \to 0$$

as $n \to \infty$ or

(17')
$$(\{(\omega-1)D+\omega Q\}u_n, Ku_n) - \lambda(\{D+\omega S\}u_n, Ku_n) \to 0$$

Since we are dealing with bounded operators the last limiting relation implies that

(18)　　　$|\lambda|^2 |((D+\omega S)u_n, Ku_n)|^2 - |([(\omega-1)D+wQ]u_n, Ku_n)|^2 \rightarrow 0$

as $n \rightarrow \infty$. However, if **A** is positive definite and $|\lambda| \geq 1$, the limiting relation (18) contradicts the inequality derived from the identity (8) for, in view of (2) and (16), the identity (8) yields the inequality

(19)　　$|((D+\omega S)u_n, Ku_n)|^2 - |([(\omega-1)D + wQ]u_n, Ku_n)|^2 \geq \omega^2 \beta \alpha$.

　　　Thus $(T-\lambda)^{-1}$ is a bounded operator defined on the closed subspace $R(T-\lambda) \subseteq H$. Moreover $R(T-\lambda) = H$ for otherwise there would exist an element v in H such that $((T-\lambda)u, v) = 0$ or $(Tu, v) = \lambda(u, v)$ for all u in H. This can be written in the form $([(\omega-1)D +\omega Q]u, (D^*+\omega S^*)^{-1}v) = \lambda(u, v)$. Since $R(K) = H$ there exists an element w in H such that $Kw = (D^* + wS^*)^{-1}v$. If we choose now $u = w$, then the last equality becomes

$$([(\omega-1)D+\omega Q]w, Kw) = \lambda((D+\omega S)w, Kw) .$$

For $|\lambda| \geq 1$ this equality contradicts the corresponding inequality (19) arising from the identity (8). This shows that $R(T-\lambda) = H$ and thus completes the proof of the theorem.

4.　　Special Cases.

　　　(i) If we take $\omega = 1$, then the theorem gives the necessary and sufficient conditions for the convergence of the generalized Seidel method, which for the case when $K = I$ and $Q = S^*$ were given in [2].

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(ii)   If we change the signs of S and Q and let
$\omega = 1$,   K = I,   and   Q = S* + F, where F is symmetric, the
theorem reduces to the result proved in [1] for the case when
the operators are finite matrices.

(iii) If D = I and Q = S*, then the theorem and the
goi-method are applicable to the operator equations of the form
(I-S-S*)u = f to which are reducible, for example, the Fredholm
integral equations of the second kind with symmetric or symmetrizable
kernels and matrix equations in which D is usually the matrix
composed of diagonal terms, S is a lower triangular and S* an
upper triangular matrix. Let us note at the end that for the
last two important classes of operator equations the set
$\Omega = \left\{ \omega,\ 0 < \omega < 2 \right\}$.

5. <u>Examples</u>.

Following the suggestion of Professor E. Isaacson we shall
illustrate here the procedure and the applicability of the
theorem to a very simple 2 x 2 algebraic equation (3), where

$$(20) \qquad\qquad A = \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} ,$$

$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a given vector, and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is to be determined.
The main problem in the effective and actual applicability of
the goi-method is the splitting of the matrix A into the form
(1), i.e.,

$$(21) \quad \begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \tau \end{pmatrix} + \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \tau_1 \end{pmatrix} + \begin{pmatrix} 2-\alpha-\alpha_1 & 4-\beta-\beta_1 \\ 1-\gamma-\gamma_1 & 5-\tau-\tau_1 \end{pmatrix},$$

so that there exists a symmetric and positive definite matrix

$$(22) \quad K = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

and a non-empty set Ω of real numbers ω such that the matrices $D, S,$ and Q satisfy the conditions of the Theorem which serve to determine the unknown parameters in (21) and (22). In what follows we shall show that there are 5-parameter families of such splittings.

Let us first note that A is necessarily K-symmetric, i.e., $KA = A^* K$, where A^* is the transpose of A. Consequently, $a_{12} = \dfrac{a_{22}-4a_{11}}{3}$. This and the condition (a) imply that

$$(23) \quad a_{11} = \frac{3\gamma + \alpha - \tau}{3\beta+4\alpha-4\tau} a_{22} , \qquad 3\beta + 4\alpha - 4\tau \neq 0 ,$$

and, therefore, that

$$(24) \quad K = \begin{pmatrix} 3\gamma + \alpha - \tau & \beta - 4\gamma \\ \beta - 4\gamma & 3\beta+4\alpha-4\tau \end{pmatrix},$$

where, without the loss of generality, we put $a_{22} = 3\beta+4\alpha-4\tau$. For K to be positive definite it is necessary and sufficient that

$$(25) \quad \begin{aligned} 3\gamma + \alpha - \tau &> 0 , \qquad 3\beta + 4\alpha - 4\tau > 0 \\ (3\gamma+\alpha-\tau)(3\beta+4\alpha-4\tau) &> (\beta - 4\gamma)^2 \end{aligned}$$

while the condition that $KS = SK$ implies that

$$(26) \qquad\qquad\qquad \gamma_1 = \beta_1$$

$$(26_1) \qquad \tau_1 = \alpha_1 + \frac{3(\beta+\alpha-\tau-\gamma)\beta_1}{\beta-4\gamma} \quad , \quad \beta - 4\gamma \neq 0 .$$

Furthermore, for the condition (b) with $G(\omega)$ given by

$$(28) \qquad G(\omega) = \frac{1}{\omega} \begin{pmatrix} 2\alpha + 2\alpha_1\omega - 2\omega & 2\beta + 2\beta_1\omega - 4\omega \\ 2\gamma + 2\beta_1\omega - \omega & 2\tau + 2\tau_1\omega - 5\omega \end{pmatrix}$$

to be satisfied we must choose the parameters so that

$$(29) \qquad \tau_1 = \frac{3(\beta+\alpha-\tau-\gamma)\beta_1 - (\beta-4\gamma)\alpha_1}{\beta_1 + 4\gamma} \quad , \quad \beta_1 + 4\gamma \neq 0 ,$$

and

$$(29_1) \qquad d = \frac{(2\alpha+2\alpha_1\omega-2\omega+2\tau+2\tau_1\omega-5\omega)}{\omega} > 0$$

$$(29_2) \qquad c = \frac{(2\alpha+2\alpha_1\omega-2\omega)(2\tau+2\tau_1\omega-5\omega)-(2\beta+2\beta_1\omega-4\omega)(2\gamma+2\beta_1\omega-\omega)}{\omega^2} > 0$$

$$(29_3) \qquad\qquad\qquad d^2 - 4c > 0 .$$

The formula (29) for τ_1 follows from the K-symmetry of G while the inequalities (29_1)-(29_3) follow from the requirement that the eigenvalues of G be real and positive for it is not hard to prove that a K-symmetric matrix is K-p.d. if and only if its eigenvalues are real and positive.

Equating (26_1) to (29) we find that

$$(30) \qquad\qquad \alpha_1 = \frac{3(\beta+\alpha-\tau-\gamma)(\beta-8\gamma-\beta_1)\beta_1}{(\beta_1+\beta)(\beta-4\gamma)}$$

$$（31） \qquad \tau_1 \; = \; \frac{6(\beta - \alpha - \tau - \gamma)\beta_1}{\beta_1 + \beta} \quad .$$

The final condition (c) will be satisfied if in addition to the inequalities $(29_1)-(29_3)$ the set $\Omega(\omega)$ is so chosen that

$$(32) \quad \det |D+\omega S| \; = \; (\alpha + \omega\alpha_1)(\tau + \omega\tau_1) - (\gamma + \omega\beta_1)(\beta + \omega\beta_1) \neq 0 \; .$$

Thus, the above discussion shows that to the matrix A given by (20) there corresponds, in general, a 5-parameter family of splittings of the form (21) depending on α, β, γ, τ, and β_1 such that if K is determined by (24) and (25) and γ_1, α_1, and τ_1 are determined by (26), (30), and (31) respectively so that the conditions $(29_1)-(29_3)$ and (32) are satisfied for $\omega \; \epsilon \; \Omega$, then the matrices D, S, G, and K fulfill all the conditions of our Theorem. Since, as can be easily checked, A is K-symmetric and has real and positive eigenvalues, it is K-p.d.. Therefore, by the Theorem, the eigenvalues $\mu_1(\omega)$ and $\mu_1(\omega)$ of $T(\omega) = (D+\omega S)^{-1}\left\{(\omega-1)D+\omega Q\right\}$ are in absolute value less than one. Hence, the goi-method converges.

A particular splitting. Let us take $\alpha = 4$, $\beta = 1$, $\tau = 2$, $\gamma = 0$, and $\beta_1 = 1$. Then by (26), (30), and (31) we get $\gamma_2 = 1$, $\alpha_1 = 0$, and $\tau_1 = 9$, respectively. Hence

$$K \; = \; \begin{pmatrix} 2 & 1 \\ 1 & 11 \end{pmatrix}, \quad D \; = \; \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}, \quad S \; = \; \begin{pmatrix} 0 & 1 \\ 1 & 9 \end{pmatrix}, \quad Q = \begin{pmatrix} -2 & 2 \\ 0 & -6 \end{pmatrix}$$

and

$$G = \frac{1}{\omega} \begin{pmatrix} 8 - 2\omega & 2 - 2\omega \\ \omega & 4 + 13\omega \end{pmatrix}, \qquad D + \omega S = \begin{pmatrix} 4 & 1 + \omega \\ \omega & 2 + 9\omega \end{pmatrix}.$$

One easily checks that for this particular splitting

$KD = D^*K, \quad KS = SK, \quad KG = G^*K, \quad KA = A^*K.$ Furthermore,

$d = \dfrac{12+11\omega}{\omega}, \quad c = \dfrac{32+94\omega-24\omega^2}{\omega^2}, \quad$ and $\quad d^2-4c = \dfrac{217\omega^2-112\omega+16}{\omega^2}.$

This shows that the conditions (29_1)-(29_3) as well as (32)

are satisfied for ω in $\Omega = \left\{ \omega, \ 0 < \omega < \dfrac{47 + \sqrt{2977}}{24} \right\}$.

In the end let us point out that the entire procedure could have been very much simplified if we had used fewer parameters in (21) and that it is immediately extendable to an n x n matrix equation.

Let us also remark that the goi-method and the Theorem are also applicable to a class of Fredholm integral equations of the second kind with a continuous symmetric or symmetrizable positive definite kernel.

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JAN 1 1 1993

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